PROBLEM SOLVING TECHNIQUES

~ Tracking Invariants ~

Get to know your numbers

$$2024 = 2^3 \cdot 11 \cdot 23$$

$$2025 = 3^4 \cdot 5^2 = (45)^2$$

Homework:

For each question think how you can change the numbers given in the question (circled blue) for the question to still make sense. (The answer may be different for different choices.)

We start with an empty room.
At each minute: - one person enters
or
- two people leave.

After 3^{2024} minutes is it possible to have $3^{1012}+2$ people in the room?

<u>Sol</u>: Let $p(t)$ be the number of people in the room at time t.

$$p(0) = 0$$

Let us look at possible initial steps:

$p(1) = 1 \quad p(2) = 2 \quad p(3) \in \{0, 3\}, \; p(4) \in \{1, 4\} \ldots$

$$p(t+1) = p(t)+1 \quad \text{or} \quad p(t)-2$$

sharp eye method:

At each step all possible outcomes can only differ by a multiple of 3.

$$(p(t)+1) - (p(t)-2) = \underline{\underline{3}}$$

At times divisible by 3 the number of people in the room is divisible by 3.
Hence, there cannot be $3^{1012}+2$ people after 3^{2024} minutes in the room.

On an island there are: 17 yellow

15 grey

13 blue cameleons

Occasionaly 2 cameleons meet.

- If two cameleons of the same color meet, their color remains unchanged.
- If two cameleons of different colors meet, they both change their color to the third color.

Can it happen that at some point all cameleons are of the same color?

Sol: Let (y_k, g_k, b_k) denote the numbers of different colored cameleons after k encounters of two different colored cameleons.

$$y_0 = 17, \quad g_0 = 15, \quad b_0 = 13$$

After an encounter: (y_k, g_k, b_k) can change into one of the following:

$$(y_{k+1}, g_{k+1}, b_{k+1}) \in \{ (y_k - 1, g_k - 1, b_k + 2), \\ (y_k - 1, g_k + 2, b_k - 1), \\ (y_k + 2, g_k - 1, b_k - 1) \}$$

$$y_{k+1} - g_{k+1} \in \{ y_k - g_k, \\ y_k - g_k - 3, \\ y_k - g_k + 3 \}$$

The remainder after division of $y_k - g_k$ by 3 remains the same.

$y_0 - g_0 = 2$, hence $y_k - g_k = 3 \cdot x_k + 2$ for some x_k for all k.

If all cameleons are of the same color at some point k, we have the following options:

$y_k - g_k = 45$ all are yellow

$y_k - g_k = -45$ all are grey

$y_k - g_k = 0$ all are blue.

Hence it is not possible for all cameleons to have the same color.

Hw: There are 3 piles with n tokens each.

In every step we randomly choose 2 piles, take a token from each of the chosen piles, and add one token to the third pile. Is it possible to end up leaving only one token.

S:

A	B	C
n	n	n

→ starting

a	b	c
$a-1$	$b-1$	$c+1$
$a-1$	$b+1$	$c-1$
$a+1$	$b-1$	$c-1$

mod 2

$(a_1 \quad b_1 \quad c_1)$

$$\left.\begin{aligned} a_1+b_1 &\equiv a+b \\ b_1+c_1 &\equiv b+c \\ a_1+c_1 &\equiv a+c \end{aligned}\right\} \text{mod } 2$$

Impossible

1	0	0
0	1	0
0	0	1

At the beginning

$$\left.\begin{aligned} a_0+b_0 &\equiv 0 \\ b_0+c_0 &\equiv 0 \\ a_0+c_0 &\equiv 0 \end{aligned}\right\} \text{mod } 2$$

PROBLEMS INVOLVING GAMES

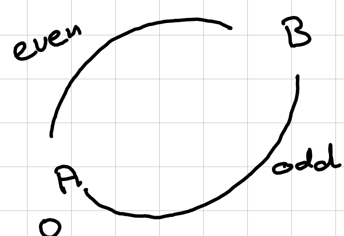
Losing position:

- a player in that state cannot win
- once a player is in a losing state the opponent can force them back into a losing state

A and B are playing a game in turns.
At the start they form a circle with 2025 other people.
At every turn they can remove one of their neighbours from the circle. The winner is the one who forces their neighbour out.
If A starts, who has the winning strategy?

Sol : Hint: 0 is an even number.
Since 2025 is an odd number there will have to be even number of people between A and B on one side of the circle and odd number on the other side.

Starting position:



The first player has a winning strategy by always removing their neighbour on the „even" side of the circle.

There are 2010 matches on the table. A and B play in turns to remove matches.
At each turn they are to remove
1, 3, 4, 5 or 7 matches.
Whoever removes the last match wins. Who has a winning strategy?

Sol:

(1) MULTIPLES OF 8 ARE LOSING POSITIONS

$\boxed{0}$ 1 $\cancel{2}$ 3 4 5 $\cancel{6}$ 7 $\boxed{8}$

- If a player has a multiple of 8, they cannot take all the matches.
- If they take x matches, the other player can take 8 − x to force them back into a multiple of 8.

(2) Numbers of the form 8k, 8k+2 are losing positions.

Once a player P has 8k + r matches on the table, and takes X matches, the opponent can take 8 − x matches to force the player into 8(k−1) + r position.

$\boxed{r=0}$ → the player P will eventually have 8 matches on the table. If they take 1 the opponent will take 7, winning the game. If they take 3, the opponent will take 5, etc.

$\boxed{r=2}$ − the player P eventually has 2 matches on the table. The only option now is to take 1, leaving the remaining 1 for the opponent to win the game.

Hw: Prove that if a player has 8k + 6 matches on the table, they have a winning strategy.